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New Value of m_μ/m_e from Muonium Hyperfine Splitting

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The complete contribution to the muonium hyperfine splitting of relative order $\alpha^3(m_e/m_\mu)\ln\alpha$ is calculated. The result amounts to 0.013 kHZ, much smaller than suggested by a previous estimate, and leads to a 2.3σ shift of the most precise value for the muon-electron mass ratio. The new value is $m_\mu/m_e = 206.7682803(44)$, with the error reduced by approximately 30%. Analogous contributions are calculated also for the positronium hyperfine splitting: $(217/90 - 17\ln 2/3)m_e(\alpha^7/\pi)\ln\alpha^{-1} \approx -0.32$ MHz; the remaining theoretical uncertainty should be well below experimental error, leaving discrepancies of 2.6σ and 3.5σ with the two most precise measurements.

Precise measurement of the ground-state muonium (μ^+e^-) hyperfine-splitting (HFS), together with the corresponding theoretical analysis, provides a stringent test of QED bound state theory, and allows a precise determination of the fundamental physical constants m_μ/m_e and α. The most precise measurement gives [**?**]:

$$\Delta\nu(\mathrm{Mu})_{\mathrm{expt.}} = 4463302.765(53)\,\mathrm{kHz}. \tag{1}$$

The theoretical prediction can be expressed as a series expansion in small parameters $\alpha \approx 1/137$ and $m_e/m_\mu \approx 1/206$. Terms involving logarithms $\ln\alpha^{-1} \approx \ln(m_\mu/m_e) \approx 5$ also appear. At leading order in α, the splitting is given by the Fermi energy [**?**]:

$$E_F = h\Delta\nu_F = \frac{16}{3}(hc)R_\infty Z^4\alpha^2\frac{m_e}{m_\mu}\left[1+\frac{m_e}{m_\mu}\right]^{-3}. \tag{2}$$

The complete splitting can be broken into the sum of terms [**?**] [**?**]:

$$\Delta\nu(\mathrm{Mu})_{\mathrm{theory}} = \Delta\nu_D + \Delta\nu_{\mathrm{rad}} + \Delta\nu_{\mathrm{rec}} + \Delta\nu_{\mathrm{r-r}} + \Delta\nu_{\mathrm{weak}} + \Delta\nu_{\mathrm{had}}. \tag{3}$$

Here D stands for Dirac, or relativistic corrections. The other terms are from radiative, recoil, radiative-recoil, weak and hadronic contributions.

Currently, theory is limited by uncalculated or imprecisely-known terms in $\Delta\nu_{\mathrm{rec}}$ and $\Delta\nu_{\mathrm{r-r}}$ of order $E_F\alpha^3(m_e/m_\mu)$, some of which are enhanced by logarithmic factors; see Table I. This paper presents a calculation of terms of order $E_F\alpha^3(m_e/m_\mu)\ln\alpha$, with results:

$$\Delta\nu_{\mathrm{rec}} = E_F\frac{(Z\alpha)^3}{\pi}\frac{m_e}{m_\mu}\ln(Z\alpha)^{-1}\left(\frac{101}{9} - 20\ln 2\right) \tag{4}$$

$$\Delta\nu_{\mathrm{r-r}} = E_F\frac{\alpha(Z\alpha)^2}{\pi}\frac{m_e}{m_\mu}\ln(Z\alpha)^{-1}\left(-\frac{431}{90} + \frac{32}{3}\ln 2 + Z^2\right). \tag{5}$$

Numerically, these contributions give $-0.042 + 0.055 = 0.013$ kHz. Previous incomplete calculations [**?**] [**?**] suggested a contribution of $-0.263(60)$ kHz. The main result of this paper is to show that in fact these contributions are not as large as the previous estimates. Remaining theoretical uncertainty is dominated by terms of order $E_F(Z\alpha)^3(m_e/m_\mu)\ln(m_\mu/m_e)$ (~ 0.06 kHz); $E_F(Z\alpha)^3(m_e/m_\mu)$ (~ 0.03 kHz); and $E_F\alpha(Z\alpha)^2(m_e/m_\mu)$ (~ 0.03 kHz). A discussion of the error due to still uncalculated terms is given at the end of the paper.

Including the complete results of Eqs.[(**??**),(**??**)] does not significantly alter the theoretical prediction for the HFS, which is in good agreement with the experimental value, Eq.(**??**). Here we simply quote the valued obtained in Ref. [**?**]:

$$\Delta\nu(\mathrm{Mu})_{\mathrm{theory}} = 4463302.67(27)\,\mathrm{kHz}. \tag{6}$$

where the error is due mainly to the measured value of m_μ/m_e [?]. Likewise, the HFS determination of α is not significantly changed [?]. However, the mass ratio as determined by the HFS becomes:

$$(m_\mu/m_e)\,[\Delta\nu(\mathrm{Mu})] = 206.7682803(33)(24)(16), \tag{7}$$

with the errors arising from uncertainty in $\Delta\nu_{\rm theory}$ due to uncalculated terms, from $\Delta\nu_{\rm expt.}$, and from the value of α, respectively. This represents a shift of 2.3σ (in terms of the previous error) and a 30% reduction in error.

The positronium HFS has also been measured precisely, though at present it's interest is for testing our knowledge of QED bound states, as opposed to determining fundamental constants. The two most precise values are due to Mills and Bearman ($\Delta\nu(P)_{\rm expt.\,1}$, Ref. [?]) and Ritter *et. al.* ($\Delta\nu(P)_{\rm expt.\,2}$, Ref. [?]):

$$\Delta\nu({\rm Ps})_{\rm expt.\,1} = 203387.5(1.6)\,{\rm MHz} \tag{8}$$

$$\Delta\nu({\rm Ps})_{\rm expt.\,2} = 203389.10(74)\,{\rm MHz}. \tag{9}$$

The theoretical expression is:

$$\Delta\nu({\rm Ps})_{\rm theory} = m_e\alpha^4\left(C_0 + C_1\frac{\alpha}{\pi} + C_{21}\alpha^2\ln\alpha^{-1} + C_{20}\alpha^2 \right. \tag{10}$$
$$\left. + C_{32}\frac{\alpha^3}{\pi}\ln^2\alpha^{-1} + C_{31}\frac{\alpha^3}{\pi}\ln\alpha^{-1} + C_{30}\frac{\alpha^3}{\pi} + \mathcal{O}(\alpha^4)\right).$$

Including the known terms through C_{20} [?] [?] [?] yields $\Delta\nu({\rm Ps})_{\alpha^2} = 203392.93\,{\rm MHz}$. Coefficient $C_{32} = -7/8$ has been known for some time [?], and in this paper we calculate:

$$C_{31} = 217/90 - 17\ln 2/3. \tag{11}$$

C_{32} and C_{31} contribute $-0.91\,{\rm MHZ}$ and $-0.32\,{\rm MHz}$ to $\Delta\nu({\rm Ps})_{\rm theory}$, respectively, bringing the theoretical prediction to:

$$\Delta\nu({\rm Ps})_{\rm theory} = 203391.70(20)\,{\rm MHz}. \tag{12}$$

The uncertainty of 0.20 MHz corresponds to a coefficient $C_{30} \approx 4$. For comparison, the numerical values of the other coefficients are: $C_0 = 0.58$, $C_1 = -1.24$, $C_{21} = 0.21$, $C_{20} = -0.39$, $C_{32} = -0.88$, $C_{31} = -1.52$. The discrepancy with experiment is significant: 2.6σ and 3.5σ for expt. 1 and expt. 2 respectively. As with the orthopositronium lifetime [?] [?], a true disagreement between experiment and QED would have important consequences.

The calculation is done in the framework of an effective Hamiltonian theory [?], taking inputs from relativistic QED field theory and from (non-relativistic) NRQED field theory [?]. The results to be derived for muonium can be translated directly to positronium by taking $m_\mu \to m_e$, and including the additional contributions from virtual e^+e^- annihilation.

The Hamiltonian can be decomposed into the sum:

$$H = H_0 + V_4 + V_5 + V_6 + V_7 \tag{13}$$

where H_0 is the unperturbed Hamiltonian for the Coulomb problem:

$$H_0 = \frac{p^2}{2m_r} - \frac{\alpha}{r}. \tag{14}$$

Potentials V_4, V_5, V_6 and V_7 give contributions to the energy of order $m\alpha^4$, $m\alpha^5$, etc. Since non-HFS operators will affect the HFS only in second- or higher-order perturbation theory, it follows that only the HFS parts of potentials V_6 and V_7 are necessary. Furthermore, any potential not contributing to S-states (in first or second order perturbation theory) may be neglected.

We will write the potentials in terms of a list of standard operators. Note that $\mathcal{O}_1 = \delta^3(r)/m_r^2$. Here $q \equiv l - k$.

$$\begin{aligned}
\langle l|\mathcal{O}_1|k\rangle &= \frac{1}{m_r^2} \\
\mathcal{O}_2 &= \frac{1}{\pi(Z\alpha)m_r^2}p^i\left(\frac{p^2}{2m_r} - \frac{Z\alpha}{r} - E\right)\ln\frac{m_r/2}{\frac{p^2}{2m_r} - \frac{Z\alpha}{r} - E}\,p^i \\
\langle l|\mathcal{O}_3|k\rangle &= \frac{1}{m_r^2}\ln\frac{q}{m_r} \\
\langle l|\mathcal{O}_4|k\rangle &= \frac{1}{m_r^2}\frac{|l\times k|^2}{q^2}
\end{aligned} \tag{15}$$

$$\langle l|\mathcal{O}_5|k\rangle = \pi(Z\alpha)\frac{|q|}{m_r}$$
$$\langle l|\mathcal{O}_6|k\rangle = \frac{q^2}{m_r^2}\ln\frac{q}{m_r}$$
$$\mathcal{O}_7 = \frac{1}{\pi(Z\alpha)}\frac{p^4}{m_r^3}$$
$$\langle l|\mathcal{O}_8|k\rangle = \frac{1}{m_r^2}\frac{|l\times k|^2}{q^4}$$
$$\langle l|\mathcal{O}_9|k\rangle = \frac{1}{m_r^2}\left(\sigma_e\cdot\sigma_\mu - \frac{3q\cdot\sigma_e q\cdot\sigma_\mu}{q^2}\right)$$

Potential V_4 is derived from tree-level NRQED diagrams containing Fermi, Darwin, Kinetic, and Dipole vertices, and contains the leading relativistic corrections:

$$V_4 = 4\pi(Z\alpha)\left\{\frac{c_F^e c_F{}^\mu}{6}\frac{m_r^2}{m_e m_\mu}\left[\sigma_e\cdot\sigma_\mu\mathcal{O}_1 + \frac{1}{2}\mathcal{O}_9\right] + \frac{1}{8}\left(c_D^e\frac{m_r^2}{m_e^2} + c_D^\mu\frac{m_r^2}{m_\mu^2}\right)\mathcal{O}_1 \right. \qquad (16)$$
$$\left. -\frac{1}{32}\left(\frac{m_r^3}{m_e^3}+\frac{m_r^3}{m_\mu^3}\right)\mathcal{O}_7 - \frac{m_r^2}{m_e m_\mu}\mathcal{O}_8\right\}.$$

Renormalization constants $c_F \approx c_D = 1 + \mathcal{O}(\alpha)$ are tabulated below. V_5 contains the leading radiative corrections:

$$V_5 = \frac{2\alpha(Z\alpha)}{3}\left(\frac{m_r^2}{m_e^2} + 2Z\frac{m_r^2}{m_e m_\mu} + Z^2\frac{m_r^2}{m_\mu^2}\right)\mathcal{O}_2 + \frac{14(Z\alpha)^2}{3}\frac{m_r^2}{m_e m_\mu}\mathcal{O}_3 \qquad (17)$$
$$+\left\{-\frac{4\alpha(Z\alpha)}{3}\left(\frac{m_r^2}{m_e^2}\ln\frac{m_r}{m_e} + Z^2\frac{m_r^2}{m_\mu^2}\ln\frac{m_r}{m_\mu}\right)\right.$$
$$\left. +(Z\alpha)^2\frac{m_r^2}{m_e m_\mu}\left[-\frac{2}{m_\mu^2 - m_e^2}\left(m_\mu^2\ln\frac{m_e}{m_r} - m_e^2\ln\frac{m_\mu}{m_r}\right) + \frac{20}{9} - \frac{2m_e m_\mu}{m_\mu^2 - m_e^2}\ln\frac{m_\mu}{m_e}\sigma_e\cdot\sigma_\mu\right] - \frac{4\alpha(Z\alpha)}{15}\frac{m_r^2}{m_e^2}\right\}\mathcal{O}_1.$$

V_4 and V_5 correctly reproduce the S-state spectrum through $\mathcal{O}(m\alpha^5)$ [?] [?]. The contribution from muon vacuum polarization is not relevant to our analysis, and has been excluded from V_5.

For V_6, only HFS terms are necessary. These again are taken directly from NRQED diagrams:

$$V_6 = 4\pi(Z\alpha)\frac{\sigma_e\cdot\sigma_\mu}{m_e m_\mu}\left\{\left[\frac{m_r^2}{m_e m_\mu}\left(\frac{c_S^e c_S^\mu}{48} + \frac{c_F^e c_F^\mu}{6} - \frac{c_F^e c_S^\mu + c_S^e c_F^\mu}{12}\right) - \frac{1}{24}\left(c_{p'p}^e c_F^\mu\frac{m_r^2}{m_e^2} + c_F^e c_{p'p}^\mu\frac{m_r^2}{m_\mu^2}\right)\right]\mathcal{O}_4 \right. \qquad (18)$$
$$\left. -\frac{1}{48}\left[c_S^e c_F^\mu\frac{m_r}{m_e} + c_F^e c_S^\mu\frac{m_r}{m_\mu}\right]\mathcal{O}_5\right\}$$

Spin-Orbit, retardation, Time-Derivative, $p'p$, and Seagull interactions have been included. Additional local operator terms, of the form $-\nabla^2\delta^3(r)$ and $\{p^2,\delta^3(r)\}$ are not shown explicitly; these terms do not generate factors of $\ln\alpha$, and so are not relevant to the present analysis [?].

The necessary renormalization constants have already been calculated [?] [?]:

$$c_F^e = 1 + a_e, \quad c_D^e = 1 + \frac{8\alpha}{3\pi}\left(-\frac{3}{8}+\frac{5}{6}\right) + 2a_e, \quad c_S^e = 1 + 2a_e, \quad c_{p'p}^e = a_e. \qquad (19)$$

Here $a_e = \alpha/2\pi + \mathcal{O}(\alpha^2)$ is the electron anomalous magnetic moment. For c^μ, m_μ and $Z^2\alpha$ are substituted for m_e and α.

Potential V_7 has no non-instantaneous HFS contribution coming from momentum $q \approx m\alpha^2$, since spin-dependent M1 multipole transitions vanish in the absence of relativistic effects, and are therefore suppressed. To fully determine V_7, we consider $\alpha^2 v^2$ contributions to the 1-loop photon-exchange scattering amplitude . The effective theory gives [?]:

$$\frac{(Z\alpha)^2}{m_e^2 m_\mu^2}q^2\sigma_e\cdot\sigma_\mu\left(-\frac{4}{9}\ln\frac{\Lambda}{\lambda} + \dots\right), \qquad (20)$$

where again analytic terms are not shown. Here $\Lambda \approx m_e$ is a UV cutoff on the photon momentum. The corresponding QED amplitude is (Figure 1):

$$\frac{(Z\alpha)^2}{m_e^2 m_\mu^2} q^2 \sigma_e \cdot \sigma_\mu \left(-\frac{4}{9} \ln \frac{\Lambda}{\lambda} + \frac{1}{6} \ln \frac{q}{\Lambda} + \dots \right). \tag{21}$$

This result has been checked both in QED Feynman gauge, and in NRQED Coulomb gauge [?]. Requiring the effective theory to match QED implies that

$$V_7 = \frac{(Z\alpha)^2}{6} \frac{m_r^2}{m_e^2 m_\mu^2} \sigma_e \cdot \sigma_\mu \mathcal{O}_6. \tag{22}$$

Contributions having a dependence on m_e, m_μ, α and Z different from Eq.(**??**) are ruled out by noticing that: (i) The non-recoil contributions are already present in V_4, V_5, V_6 (as we will soon verify), so that V_7 contains no non-recoil piece; (ii) masses can enter only as inverse powers $1/m_e$ and $1/m_\mu$, and in particular not as $1/(m_e + m_\mu)$. This latter result can be seen clearly using time ordered perturbation theory in NRQED: the NRQED vertices are all homogeneous in the masses, leaving only the energy denominators to consider; however, the energy denominators will all have the form $1/(|q| + p_1^2/m_e + p_2^2/m_\mu)$, with photon momentum q and particle momenta p_1, p_2. (Contributions which are not simply iterations of lower-order potentials must have at least one photon in each intermediate state.) Such an expression, for $q \approx p_1 \approx p_2 \approx m\alpha$, can be expanded in powers of $p_1^2/m_e|q|$, $p_2^2/m_\mu|q|$—again homogeneous in the masses. Using (i) and (ii), the only possible parameter dependence which is symmetric in m_e and m_μ is that of Eq.(**??**).

Having completed the specification of the Hamiltonian—Eqs.[(**??**),(**??**),(**??**),(**??**),(**??**),(**??**)]—we now use the usual expressions from Rayleigh-Schrödinger perturbation theory to solve for the energy shift:

$$\Delta E = \langle V_7 \rangle + \left\langle (V_4 + V_5) \tilde{G} (V_4 + V_5) \right\rangle + \langle V_4 \rangle \left\langle \frac{\partial V_5}{\partial E} \right\rangle, \tag{23}$$

where $\tilde{G}$ is the Coulomb Green's function with ground state pole removed, and $\langle V \rangle \equiv \langle \psi_0 | V | \psi_0 \rangle$ is the expectation value of V in the ground state of the unperturbed H_0, Eq.(**??**). The logarithmic contributions of the necessary matrix elements are:

$$\frac{\langle \mathcal{O}_i \rangle}{\langle \delta^3(r) \rangle} \to (Z\alpha)^2 \ln(Z\alpha)^{-1} \begin{cases} 2, & i = 4 \\ 8, & i = 5 \\ 12, & i = 6 \end{cases} \tag{24}$$

$$\left(2 \frac{\langle \mathcal{O}_i \tilde{G} \delta^3(r) \rangle}{\langle \delta^3(r) \rangle} + \langle \frac{\partial \mathcal{O}_i}{\partial E} \rangle \right) \to \frac{(Z\alpha)}{\pi} \ln(Z\alpha)^{-1} \times \begin{cases} -2, & i = 1 \\ -4\ln(Z\alpha)^{-1} + 6 - 8\ln 2, & i = 2 \\ \ln(Z\alpha)^{-1} + 1 - 2\ln 2, & i = 3 \\ -16, & i = 7 \\ -1, & i = 8 \end{cases} \tag{25}$$

$$\frac{\langle \mathcal{O}_9 \tilde{G} \mathcal{O}_9 \rangle}{\langle \delta^3(r) \rangle} \to \frac{10}{m_r^2} \frac{(Z\alpha)}{\pi} \ln(Z\alpha)^{-1} \tag{26}$$

where the arrows signify that only logarithmic corrections, and in the case of $\langle \mathcal{O}_9 \tilde{G} \mathcal{O}_9 \rangle$, only the HFS part, are shown [?]. The pure recoil result at order $E_F (Z\alpha)^3 (m_e/m_\mu)$ contains the previously known $\ln^2(Z\alpha)$ and $\ln(Z\alpha)\ln(m_\mu/m_e)$ contributions [?] [?] [?]; the new $\ln(Z\alpha)$ term is shown in Eq.(**??**) [?]. For radiative corrections, the non-recoil $\ln^2(Z\alpha)$ and $\ln(Z\alpha)$ terms, and the recoil $(m_e/m_\mu)\ln^2(Z\alpha)$ term [?], agree with previous calculations. A part of the recoil single-logarithm corresponding to reduced mass factor $m_r^2/m_e m_\mu \approx (1 - 2m_e/m_\mu)$ was included previously [?] [?]; the complete contribution is given in Eq.(**??**).

For positronium, there are additional interactions due to virtual annihilation of the electron and positron. However, because the annihilation is a hard process, no non-analytic terms can arise, so that for $\ln\alpha$ contributions, only second order perturbations involving V_4 and V_5 need be considered:

$$\delta V_4 = \frac{\pi\alpha}{2} \left(\frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4} \right) \mathcal{O}_1, \quad \delta V_5 = \alpha^2 \left[\left(-\frac{22}{9} \right) \left(\frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4} \right) + (-1 + \ln 2) \left(\frac{1}{4} - \frac{\sigma_e \cdot \sigma_\mu}{4} \right) \right] \mathcal{O}_1. \tag{27}$$

δV_4 gives the leading contribution from 1-photon annihilation. The first and second terms of δV_5 come from radiative corrections to δV_4, and from 2-photon virtual annihilation, respectively. $\mathcal{O}(m\alpha^7 \ln\alpha)$ contributions from these annihilation operators are:

$$\Delta\nu_A(\alpha^7 \ln\alpha) = m\frac{\alpha^7}{\pi}\ln\alpha^{-1}\left[-\frac{3}{8}\ln\alpha^{-1} + \frac{2261}{1080} - 3\ln 2\right]. \tag{28}$$

The non-annihilation contributions for positronium are obtained by taking the limit $m_\mu \to m_e$ in the muonium analysis (making no expansion in m_e/m_μ); the combined result is given in Eq.(??);

The previous most significant sources of error in the muonium HFS were $\Delta\nu_{\mathrm{r-r}}$ (0.104 kHz) and $\Delta\nu_{\mathrm{rec}}$ (0.060 kHz) [?]; all other uncertainties are estimated below 0.010 kHz [?] [?]. By calculating the $\mathcal{O}(E_F\alpha^3(m_e/m_\mu)\ln\alpha)$ contribution to $\Delta\nu_{\mathrm{r-r}}$, the uncertainty in this quantity should be reduced by a factor $\sim \ln\alpha^{-1} \approx 5$; in fact, since there are still uncalculated terms at $\mathcal{O}(E_F\alpha^2(Z\alpha)(m_e/m_\mu)\ln(m_\mu/m_e))$ [?] and $\mathcal{O}(E_F\alpha(Z\alpha)^2 m_e/m_\mu)$ [?], we take this uncertainty as 0.040 kHz. The uncertainty in $\Delta\nu_{\mathrm{rec}}$ should remain approximately the same, since it is dominated by the still uncalculated terms of order $\mathcal{O}(E_F(Z\alpha)^3(m_e/m_\mu)\ln(m_\mu/m_e))$ [?] and $\mathcal{O}(E_F(Z\alpha)^3(m_e/m_\mu))$ [?]. Thus we take 0.070 kHz as an estimate of the total remaining theoretical error.

In the final stages of the calculation, I received word from K. Melnikov and A. Yelkhovsky that they have also performed the calculation of $\alpha^3\ln\alpha$ terms, in a dimensional regularization approach [?]. After a detailed comparison, we agree fully on the contributions in both muonium and positronium. The agreement found in different formalisms in two independent calculations lends strong support to the correctness of the results.

This work was motivated in part by, and is an extension of, Ref. [?]. Many ideas used in the calculation originated with G. P. Lepage, who I thank for continued insights and encouragement during the present work. Thanks are also due to P. Labelle, and to K. Melnikov and A. Yelkhovsky for useful conversations.

$\times\frac{E_F}{\pi}\frac{m_e}{m_\mu}$	Ref. [?]	present paper
$(Z\alpha)^3\ln^2 Z\alpha$	-0.043	
$(Z\alpha)^3\ln Z\alpha\ln(m_\mu/m_e)$	-0.210	
$(Z\alpha)^3\ln Z\alpha$	$-0.257(55)$	-0.034
$(Z\alpha)^3\ln(m_\mu/m_e)$	—	-0.035 (*)
$(Z\alpha)^3$	0.107(30)	
$\alpha(Z\alpha)^2\ln^2 Z\alpha$	0.344	
$\alpha(Z\alpha)^2\ln Z\alpha$	-0.008 (*)	0.034
$\alpha(Z\alpha)^2$	$-0.107(30)$	
$Z^2\alpha(Z\alpha)^2\ln Z\alpha$	—	0.013
$\alpha^2(Z\alpha)\ln^3(m_\mu/m_e)$	-0.055	
$\alpha^2(Z\alpha)\ln^2(m_\mu/m_e)$	0.010	
$\alpha^2(Z\alpha)\ln(m_\mu/m_e)$	0.009 (*)	
$\alpha^2(Z\alpha)$	—	

TABLE I. Contributions of order $E_F\alpha^3(m_e/m_\mu)$ to the muonium HFS. The second column lists the contributions used in Ref.[3]; the third column gives new or modified values from the present paper. Asterisks denote partial results.

† rjh@mail.lns.cornell.edu
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[19] The cutoff-dependent part of c_D from Ref. [**?**], Eq.(56), has been absorbed into V_5.
[20] The analysis is similar to the calculation of c_{q^2} in section C of Ref. [**?**].
[21] For the NRQED case, a spin-dependent "seagull" interaction with two magnetic photons is necessary. The Feynman rule is: $ie^2/8m^2(p_0 - q_0)\epsilon_{ijk}\sigma_i$, where (p, j) and (q, k) are the two incoming photon momenta and polarization directions. This vertex, coming from the $\sigma \cdot E \times D$ part of the NRQED Lagrangian, was not necessary in the non-recoil calculation performed in Ref. [**?**].
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[23] S. G. Karshenboim, Z. Phys. **36** 11 (1996).
[24] The large difference between Eq.(**??**) and previous recoil results can be traced to the treatment of the operator $\mathcal{O}_3$ in V_5. In Refs. [**?**] [**?**], the single-logarithm contribution at $\mathcal{O}(E_F(Z\alpha)^3(m_e/m_\mu)\ln(Z\alpha))$ is that which is obtained from a delta-function potential giving the same contribution to the ground-state energy. The contribution from potential V_7 accounts for the remaining discrepancy.
[25] The uncertainty of 0.023 kHz for the coefficient $D^{(4)}$ in $\Delta\nu_{\rm rad}$ of Ref. [**?**] is actually dominated by a radiative-recoil piece which was included in the quantity $[-86(18)]$, Eq.(D6) *ibid.*.
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[28] The constant terms at $\mathcal{O}(E_F\alpha(Z\alpha)^2 m_e/m_\mu)$ and $\mathcal{O}(E_F(Z\alpha)^3 m_e/m_\mu)$ were estimated in the NRQED framework [**?**] [**?**].
[29] The $(Z\alpha)^3(m_e/m_\mu)\ln(m_\mu/m_e)$ contribution can be expressed as: $E_F[(Z\alpha)^3/\pi](m_e/m_\mu)\ln(m_\mu/m_e)[-3\ln(Z\alpha)^{-1} + 3\ln 2 - 9/2 + 2C/3]$, where parameter C is determined by the coefficient of $(1/m_\mu^2)\ln(m_\mu/m_e)$ in the HFS part of the 3-loop threshold scattering amplitude: $\sigma_e \cdot \sigma_\mu[(Z\alpha)^4/m_\mu^2]\ln(m_\mu/m_e)[-8(2\ln 2 + 1)(m_e^2/\lambda^2) - 2\ln(m_e/\lambda) + C + \mathcal{O}(\lambda/m_e)]$. This result was derived directly from results for positronium HFS in P. Labelle, Cornell Univerity Ph.D. Thesis (1994), and has also been checked using the techniques of Ref. [**?**]. The gauge invariant contribution, $3\ln 2 - 9/2$ would contribute -0.035 kHz to the HFS. We do not include this in the final numerical values, but simply note that the size of this term does not contradict our suggested error estimates.
[30] K. Melnikov and A. Yelkhovsky, to be published.